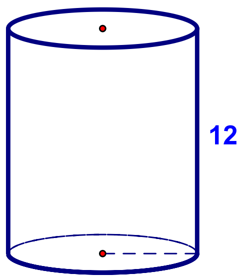
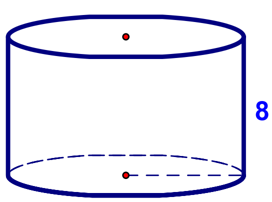
Surface Area of Circular Solids - Lesson 12-3

Warmup!

Gandalf took a piece of paper 8 by 12 inches and made a cylindrical "can" out of it. He then put a top and bottom on the can. Compute the surface area of the can.



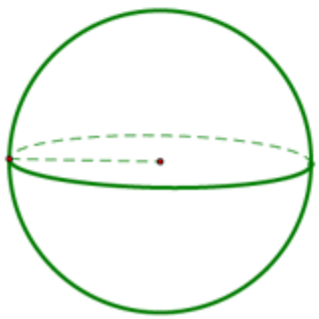
Frodo took a piece of paper 8 by 12 inches and did the same thing as Gandalf. Compute the surface area of Frodo's cylindrical can.



Today we're going to talk about the surface area of circular solids. It's a long lesson, so get comfy! We'll start by defining *spheres*, *hemispheres*, *cylinders*, and *cones*:

A *sphere* is the set of all points in space at a given distance from a given point.

A *hemisphere* is half a sphere.

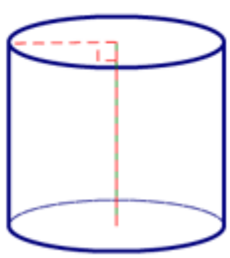


Sphere

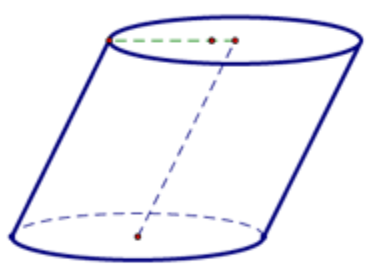
Hemisphere

A *cylinder* is a solid composed of two congruent circles in parallel planes, their interiors, and all the line segments parallel to the axis with endpoints on the two circles. The axis of a cylinder is the segment connecting the centers of the circles. The circles and their interiors are the bases, and the radius of the cylinder is the radius of a base.

If the axis of a cylinder is perpendicular to the bases, then the cylinder is *right*. If it is not, then the cylinder is *oblique*.



Right cylinder

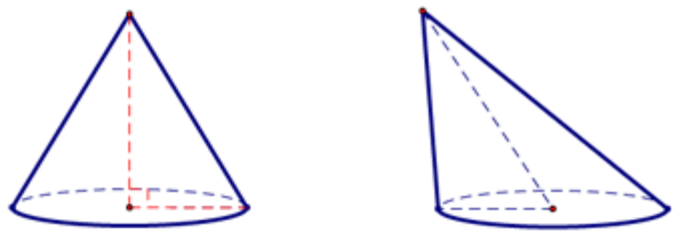


Oblique cylinder

The *axis of a cylinder* is the segment connecting the centers of the bases.

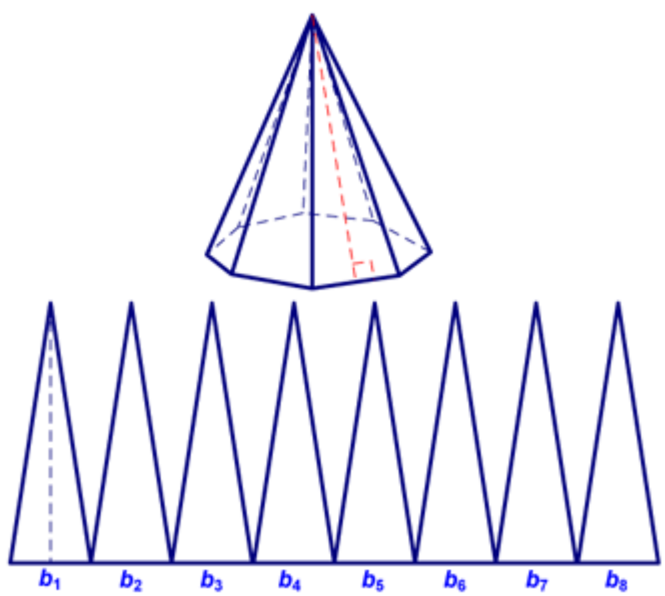
A ***cone*** is a solid composed of a circle, its interior, a given point not on the plane of the circle, and all the segments from the point to the circle.

The circle and its interior make up the ***base*** of the cone. The ***radius*** of the cone is the radius of the base. The ***altitude*** of a cone is the perpendicular segment from the vertex to the plane of the base. The ***height*** of a cone is the length of the altitude. If the line segment connecting the vertex of a cone with the center of its base is perpendicular to the base, then it is a ***right cone***.



Right Cone **Oblique Cone**

Next, we can derive the formula for the surface area of a cone. Let's start by thinking about how to find the lateral area of an octagonal pyramid:



$$\text{Lateral SA} = \frac{1}{2}b_1 l + \frac{1}{2}b_2 l + \frac{1}{2}b_3 l + \frac{1}{2}b_4 l + \frac{1}{2}b_5 l + \frac{1}{2}b_6 l + \frac{1}{2}b_7 l + \frac{1}{2}b_8 l$$

$$= \frac{1}{2}l\,(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8) = \frac{1}{2}l\,(\text{perimeter of base})$$

Now, think about how this would work if we increase the number of sides of the base to 10, 20, 100, or 1,000,000...it would get closer and closer to the lateral surface area of a cone, right? The perimeter of the base would get closer and closer to the circumference of the circular base, right? From this, we can come up with the following theorem:

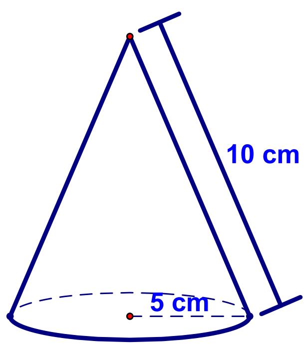
Theorem 112

The lateral area of a cone is equal to one-half the product of the slant height and the circumference of the base.

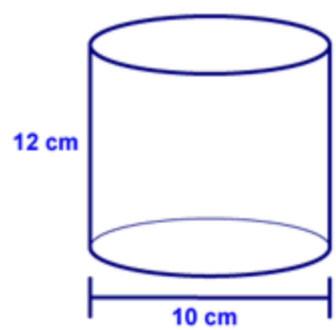
$$A_{L.S.} = \frac{1}{2}Cl = \pi rl$$

Where C is the circumference of the base, l is the slant height, and r is the radius of the base.

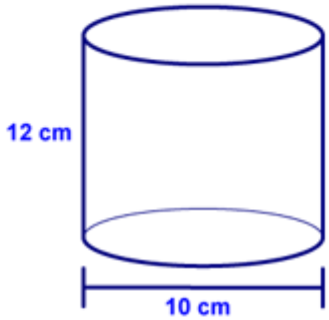
OK…you try the following example using this new theorem. Find the total surface area of the cone:

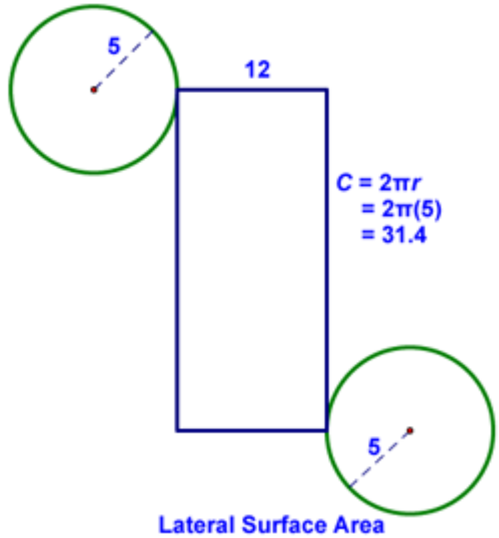


Now, let’s look at finding the surface area of the following cylinder:



In order to do this, you can "unwrap" the cylinder to look like:





From this, you should be able to calculate the surface area. The only complicated part is to recognize that the "width" (assuming that the height is 12) of the rectangle that forms the lateral surface area is the circumference of the circular bases.

This problem can be summarized in the following theorem:

Theorem 111

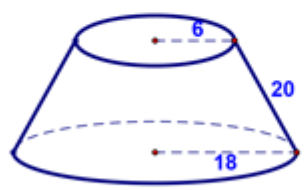
The lateral area of a cylinder is equal to the product of the height and the circumference of the base.

$$A_{L.S.} = CH = 2\pi rH$$

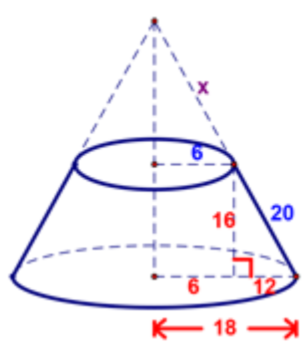
Where C is the circumference of the base, H is the height of the cylinder, and r is the radius of the base.

Next, we're going to talk about how to find the surface area of a *frustum*, which I liken to a gear. A frustum is the solid left when you take away the top part of a cone (see the diagram below).

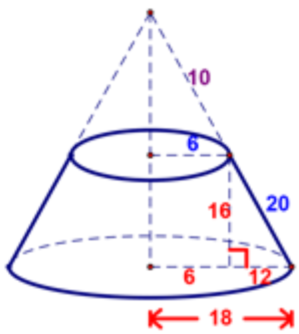
A frustum of a right cone has a slant height of 20 inches The radius of the bottom base is 18 inches and the radius of the top base is 6 inches. Find the total surface area of the frustum.



We should start by showing how this frustum was made by taking away the top part of a cone. When we draw this in, we have similar triangles and are able to determine a number of lengths based on this:



We finding the value of x, the slant height of the cone cut out to form the frustum. This is done using a proportion we can set up based on the similar triangles:

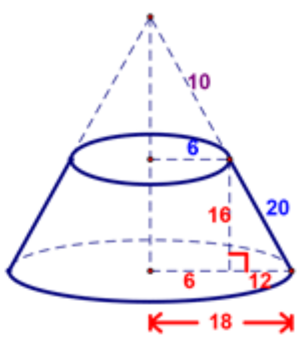


Now, the small Δ ~ to the large Δ by AA~, so

$$\frac{6}{18} = \frac{x}{x+20}$$

$$\Rightarrow x = 10$$

From here, we are able to calculate the surface area:



Now, the small Δ ~ to the large Δ by AA~, so

$$\frac{6}{18} = \frac{x}{x+20}$$

$$\Rightarrow x = 10$$

$$A_{T.S.} = A_{L.S.} + A_B$$

$$= (\pi RL - \pi rl) + (\pi R^2 + \pi r^2)$$

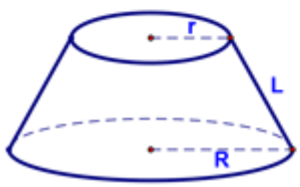
$$= (\pi(18)(30) - \pi(6)(10)) + (324\pi + 36\pi)$$

$$= 480\pi + 360\pi = 840\pi \text{ in}^2$$

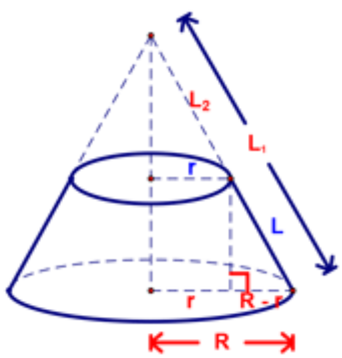
Before moving on from frustums, I want to show you a method for determining the lateral surface area that is sometimes simpler than subtracting the lateral area of the smaller cone from the lateral area of the larger cone. The derivation shown below is a little complex, but the end result is nice...

We start with a frustum labeled as shown:

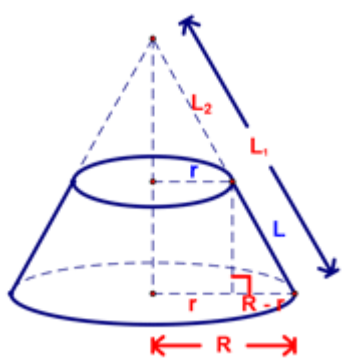
Derive a formula for the lateral surface of a frustum given the radii of the circles and the slant height of the frustum.



Next, we draw in the "missing cone" and do some labeling:



From here, we use similar triangles to find values for L_1 and L_2:



Using similar triangles, we can set up the following proportion:

$$\frac{L_1}{R} = \frac{L}{R - r} \Rightarrow L_1 = \frac{RL}{R - r}$$

But, $L_2 = L_1 - L$, so substituting our new value for L_1, we get:

$$L_2 = \frac{RL}{R - r} - L = \frac{RL - (R - r)L}{R - r} = \frac{RL - RL + rL}{R - r} = \frac{rL}{R - r}$$

Now we use the same approach as in the previous example, do some simplification and derive a new formula:

$$\text{Now, } A_{L.S.} = \pi RL_1 - \pi rL_2 = \pi R\left(\frac{RL}{R - r}\right) - \pi r\left(\frac{rL}{R - r}\right)$$

$$= \frac{\pi R^2 L}{R - r} - \frac{\pi r^2 L}{R - r} = \frac{\pi R^2 L - \pi r^2 L}{R - r}$$

$$= \frac{\pi (R^2 - r^2)L}{R - r} = \frac{\pi (R + r)(R - r)L}{R - r} = \pi (R + r)L$$

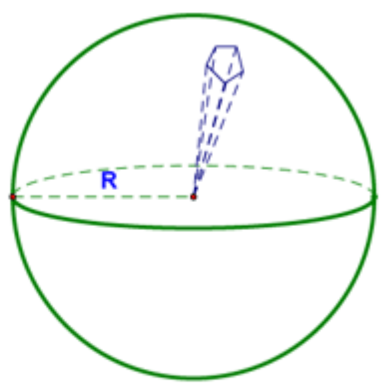
So, to find the lateral area of the first example we did, all we need to do using this formula is:

$$A_{L.S.} = \pi (R + r)L$$

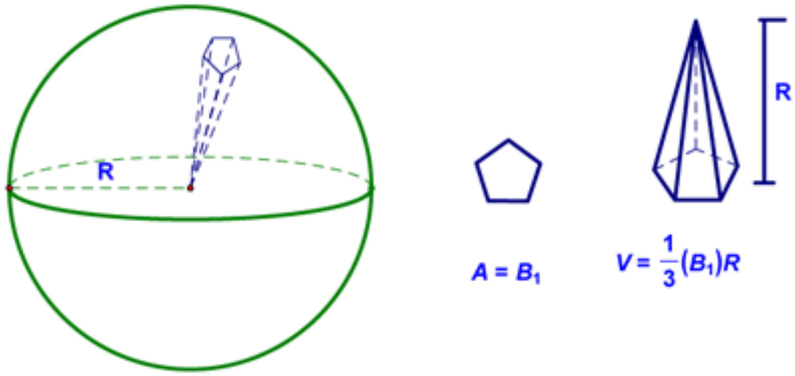
$$= \pi (18 + 6)20 = 480\pi$$

You can choose to use this formula or not...it's up to you. You can always just subtract the lateral area of the smaller cone from the lateral area of the larger one to find the lateral area of the frustum, as we did in our first example.

OK, now we're going to talk about how to derive the formula for the surface area of a sphere. We start with a sphere - imagine breaking it up into 1000 "sort-of" polygons:



Each of these is a pyramid with a volume as shown below:



Note that the surface area of the sphere is the sum of the area of the 1000 bases of the pyramids:

$$\Rightarrow A_{Surface} = B_1 + B_2 + B_3 + \ldots + B_{1000}$$

Also note that the volume of the sphere is the sum of the volumes of the 1000 pyramids (from which we can factor out the surface area):

$$\Rightarrow V = \frac{1}{3}(B_1)R + \frac{1}{3}(B_2)R + \frac{1}{3}(B_3)R + \ldots + \frac{1}{3}(B_{1000})R$$
$$= \frac{1}{3}R(B_1 + B_2 + B_3 + \ldots + B_{1000})$$

Now, using the formula for the volume of a sphere, we are able to derive a formula for the surface area:

$$\Rightarrow \frac{4}{3}\pi R^3 = \frac{1}{3}RA_{Surface}$$
$$\Rightarrow A_{Surface} = 4\pi R^2$$

This is summarized by the following postulate:

Postulate:

The surface area (S) of a sphere with radius r is given by the formula $S = 4\pi r^2$.

You should be able to get the formula for the lateral surface area and total surface area of a hemisphere from this…

Whew…that was a big day!!