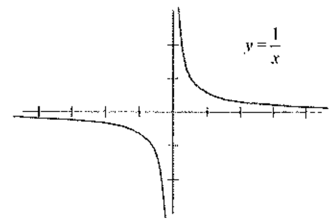
Section 2.6 – Rational Functions

Rational Functions are ratios (quotients) of polynomials, written in the form $f(x)=\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials, and $D(x)$ does not equal zero.

The parent function for rational functions is $f(x)=\frac{1}{x}$.



The **domain** of a rational function is the set of all real numbers EXCEPT when the denominator equals zero.

Every rational function is **continuous** on its domain (you won't pick up your pencil), but you may have vertical asymptotes that break the graph into distinct sections, or a hole that causes a temporary disruption.

Examples:

Identify which of the following are rational functions.
For those that are not, explain why not. For those that are rational, state the domain.

1) $f(x)=\frac{3x}{x^2+1}$ yes

$D: \mathbb{R}$

2) $g(x)=\frac{x^3+2}{\sqrt{x}}$ no!

$x^{\frac{1}{2}}$ must be an integer!

3) $h(x)=\frac{\sqrt{25x^4+4}}{7x^3+6x}$ NO!

no radicals!

4) $k(x)=\frac{3x-5}{6x-2x^3}$ yes

$6x-2x^3 \neq 0$

$2x(3-x^2) \neq 0$

$D: \{x \neq 0, \pm\sqrt{3}\}$

5) $p(x)=\frac{-9}{x+8}$ yes

$D: \{x \neq -8\}$

6) $q(x)=3x^2-x+4$ yes

$D: \mathbb{R}$

(really just a polynomial)

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Graphical Features of a Rational Function found ALGEBRAICALLY:

The biggest thing in terms of graphing to watch out for is a HOLE- it is not easy to see on a graph. To identify hole in your equation, you can do one of two things:

1) Factor the originals- if the top and the bottom have the same factor, then there is a hole there
2) Look at the TABLE in your calculator- if there is an x-value that does not have a y-value, then there is a hole there.

Example: $f(x)=\frac{2x^2-4x-6}{x^2-9x+18} = \frac{2(x^2-2x-3)}{(x-6)(x-3)} = \frac{2(x-3)(x+1)}{(x-6)(x-3)}$

hole at $x=3$!!

VERTICAL ASYMPTOTES:

Vertical asymptotes are vertical lines, and occur when the denominator of the function equals zero.

*** The only exception is when that value of x makes both the numerator AND the denominator zero- then it is not an asymptote, but instead a HOLE in the function.

Examples: Find the vertical asymptotes:

1) $f(x)=\frac{3x^2+6x+3}{x^2-4x-12}$

$= \frac{3(x+1)(x+1)}{(x-6)(x+2)}$

VA : $x=6, x=-2$

2) $g(x)=\frac{x-5}{x^3+4x^2}$

$= \frac{x-5}{x^2(x+4)}$

VA: $x=0, x=-4$

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X-INTERCEPTS:
X-intercepts occur when the function equals zero. The quick way to discover this is to set the NUMERATOR equal to zero.

*** The only exception is when that value of x makes both the numerator and the denominator zero- again, this will create a HOLE instead of a zero.

Y-INTERCEPT:
The y-intercept is easy- just plug x = 0 into the equation and solve.

Examples: Find the x and y-intercepts:

1) $f(x)=\frac{3x^2+6x+3}{x^2-4x-12}=\frac{3(x+1)(x+1)}{(x-6)(x+2)}$

x-int: $(-1,0)$

y-int: $y=\frac{3}{-12}=-\frac{1}{4}$

$(0,-1/4)$

2) $g(x)=\frac{x-5}{x^3+4x^2}=\frac{x-5}{x^2(x+4)}$

x-int: $(5,0)$

y-int: $-\frac{5}{0} \Rightarrow$ none!

HORIZONTAL ASYMPTOTES:

If the leading exponent on the bottom is greater than the leading exponent on the top, then the horizontal asymptote is y = 0.

If the leading exponent on the bottom is equal to the leading exponent on the top, then the horizontal asymptote is the fraction y = $\frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of the numerator and denominator.

If the leading exponent on the bottom is less than the leading exponent on the top, then there is no horizontal asymptote ($y\to\infty$ or $y\to-\infty$).

Examples: Find the horizontal asymptotes:

1) $f(x)=\frac{3x^2+6x+3}{x^2-4x-12}$

HA: $y=3$

2) $g(x)=\frac{x-5}{x^3+4x^2}$

HA: $y=0$

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Let's put it all together! Based on the information gathered above, let's draw a sketch of the function based on the horizontal and vertical asymptotes and the x and y-intercepts:

Let's summarize what we found above first:

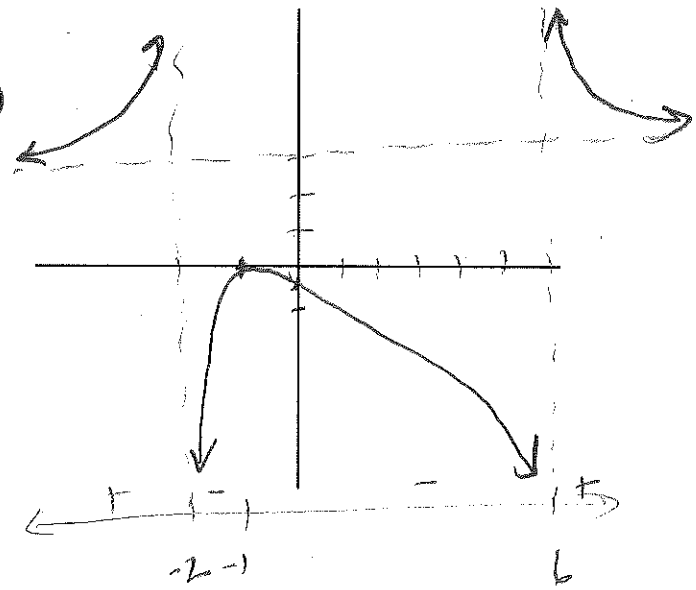
For #1: $f(x)=\dfrac{3x^2+6x+3}{x^2-4x-12} = \dfrac{3(x+1)(x+1)}{(x-6)(x+2)}$

HA: $y=3$

VA: $x=6, x=-2$

x-int: $(-1,0)$

y-int: $(0,-1/4)$



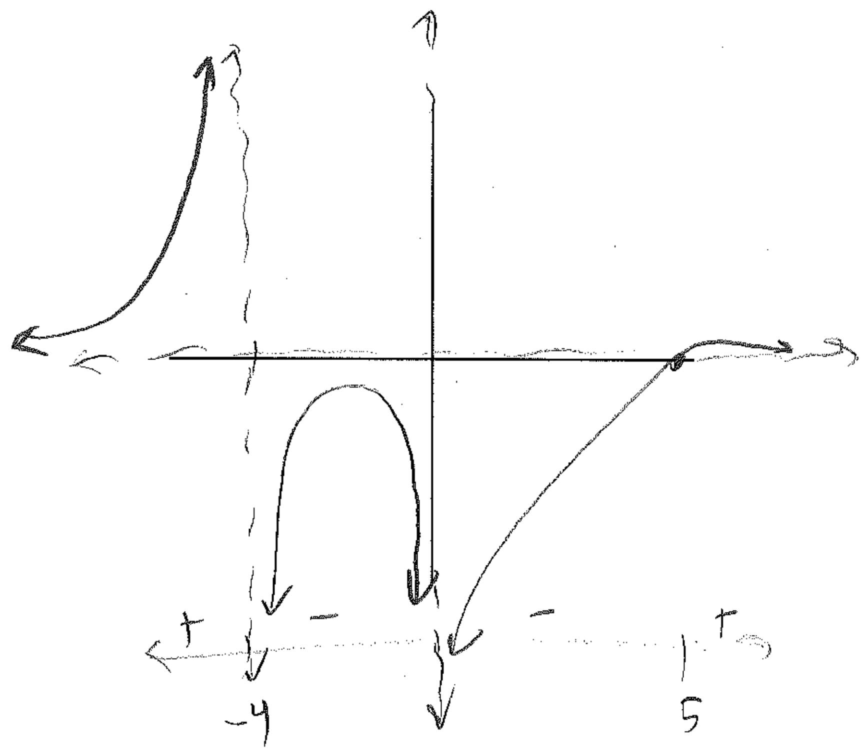
For #2: $g(x)=\dfrac{x-5}{x^3+4x^2} = \dfrac{x-5}{x^2(x+4)}$

HA: $y=0$

VA: $x=0, x=-4$

x-int: $(5,0)$

y-int: none



Slant Asymptotes

If the leading exponent on the bottom is less than the leading exponent on the top, then there is no horizontal asymptote, because there is a SLANT asymptote.

To find the equation of the slant asymptote, use long (or synthetic) division!

Example: Find the asymptotes of $f(x)=\dfrac{x^2-x}{x+1}$

$$\begin{array}{r|rrr} -1 & 1 & -1 & 0 \\ & & -1 & 2 \\ \hline & 1 & -2 & 2 \end{array}$$

remainder not needed

The equation of the slant asymptote is: $y = x - 2$

Now, let's practice graphing too!

1) $f(x)=\dfrac{2x^2-3x-5}{x-2} = \dfrac{(2x-5)(x+1)}{x-2}$

slant!

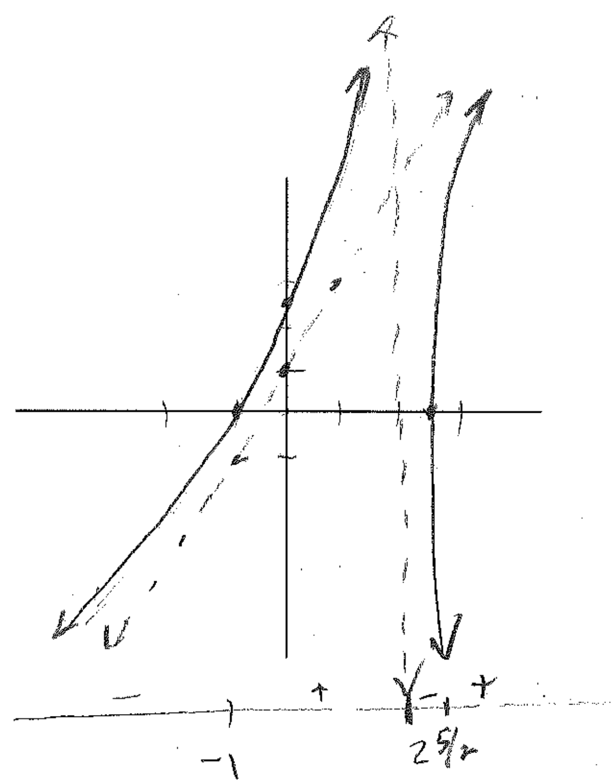
SA: $y = 2x + 1$

VA: $x = 2$

x-int: $(5/2, 0), (-1, 0)$

y-int: $(0, 5/2)$

$$\begin{array}{r|rrr} 2 & 2 & -3 & -5 \\ & & 4 & 2 \\ \hline & 2 & 1 & -3 \end{array}$$



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Put it all together! Find the domain, holes (if any) asymptotes, and intercepts, then graph:

1) $f(x)=\dfrac{4x^2-4x-8}{2x^2+6x-20} = \dfrac{4(x^2-x-2)}{2(x^2+3x-10)} = \dfrac{4(x-2)(x+1)}{2(x+5)(x-2)}$

Domain: $\{x = -5, x \neq 2\}$

Hole: $x = 2$

VA: $x = -5$

SA or HA: $y = \frac{4}{2} = 2$

x-int: $(-1, 0)$

y-int: $(0, 2/5)$