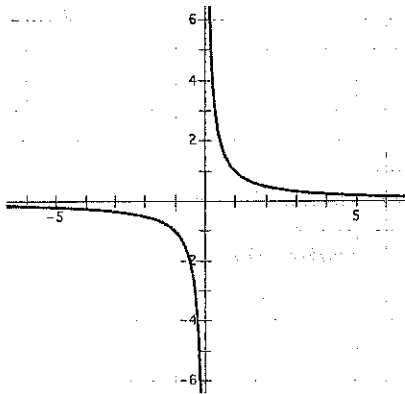


## Horizontal and Vertical Asymptotes

Horizontal Asymptotes:

Graph of  $f(x) = \frac{1}{x}$ 

$$\lim_{x \rightarrow \infty} \frac{1}{x} = \underline{0} \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = \underline{0}$$

Limits as  $x$  goes to INFINITY will create a horizontal asymptote!

The horizontal line  $y=0$  is a horizontal asymptote of the graph of the function if either  $\lim_{x \rightarrow \infty} f(x) = 0$  or  $\lim_{x \rightarrow -\infty} f(x) = 0$ . Horizontal asymptotes show the end behavior of a function as  $x$  approaches  $\pm\infty$ .

Since there are two ways to create a horizontal asymptote, a graph can have up to two (but this is unusual- most will only have one, and all rational functions only have one).

RULES (these should look familiar):

For a rational function (a polynomial function over a polynomial function):

- If the degree of the denominator is bigger then there is a horizontal asymptote at  $y=0$ .
- If the degree of the numerator is bigger then there is *slant* asymptote.
- If the degrees are equal, then there is horizontal asymptote at  $y = \frac{a_n}{b_n}$ ,

where  $a_n$  and  $b_n$  are the leading coefficients of the numerator and the denominator.

## Limits involving Infinity

Why does this work?

Algebraically determine the  $\lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{2x^2 + 4x - 7}$

Divide every term by the *highest* power in the denominator:

The limit can now be expressed as  $\lim_{x \rightarrow \infty} \frac{\frac{5x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{4x}{x^2} - \frac{7}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{3}{x} + \frac{1}{x^2}}{2 + \frac{4}{x} - \frac{7}{x^2}} = \frac{5 - 0 + 0}{2 + 0 - 0} = \frac{5}{2}$

For non-rational functions (exponential, logarithmic, trig), we use the graph (visual limits) to determine limits/asymptotes or we may intuitively know. We'll do this on another day!

## Relating Horizontal Asymptotes to Limits

Determine the value of the limits:

$\lim_{x \rightarrow \infty} \frac{3x^3 - x + 1}{x + 3} = \lim_{x \rightarrow \infty} \frac{3x^2 - 1 + \frac{1}{x}}{1 + \frac{3}{x}} = \infty$	$\lim_{x \rightarrow \infty} \frac{1 - 7x^2}{x + 5} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - 7x}{1 + \frac{5}{x}} = -\infty$
$\lim_{x \rightarrow \infty} \frac{7x^2 + 1}{-x + 5} = \lim_{x \rightarrow \infty} \frac{-7x + \frac{1}{x}}{1 + \frac{5}{x}} = -\infty$	$\lim_{x \rightarrow \infty} \frac{5x}{x^2 + 1} = 0$
$\lim_{x \rightarrow \infty} \frac{3x^2}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{3}{2 + \frac{1}{x^2}} = \frac{3}{2}$	$\lim_{x \rightarrow \infty} \frac{3x}{x^3 - 1} = 0$
$\lim_{x \rightarrow \infty} \frac{8x + 1}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{8}{x} + \frac{1}{x^2}}{1} = 0$	$\lim_{x \rightarrow \infty} \frac{-8x + 1}{x^2} = 0$

## Limits involving Infinity

### Vertical Asymptotes:

You already know that if you want to find a vertical asymptote in a rational function, look for values that would make denominator equal to ZERO **but** ones that don't cancel with a term from the numerator (when a term cancels out from the numerator and denominator then it makes a HOLE). So here is how it looks with limits:

→ If  $\lim_{x \rightarrow a} f(x) = \frac{0}{0}$  then there is a HOLE at  $x = a$

→ If  $\lim_{x \rightarrow a} f(x) = \frac{n}{0}$ , where  $n$  is a real number, then there is a vertical asymptote at  $x = a$

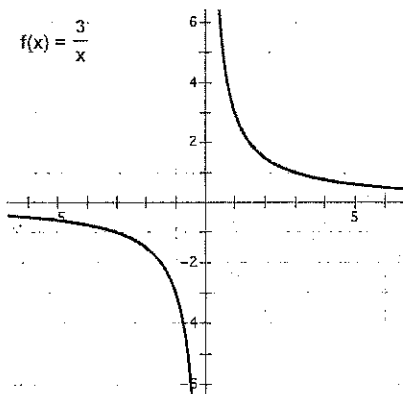
The line  $x = a$  is a vertical asymptote of the graph of  $f(x)$  if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^-} f(x) = \pm\infty$$

$\lim_{x \rightarrow a^+}$  means you are looking at the RIGHT side of the  $x$ -value "a"

$\lim_{x \rightarrow a^-}$  means that you are looking at the LEFT side of the  $x$ -value "a"

### Visually:



$$\lim_{x \rightarrow 0^+} \frac{3}{x} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{3}{x} = -\infty$$

Therefore,  $\lim_{x \rightarrow 0} \frac{3}{x} =$

DNE

does not exist!

## Limits involving Infinity

Example: Determine the value of  $\lim_{x \rightarrow 3} \frac{x^2+1}{3+x}$  if it exists.

Must first consider  $\lim_{x \rightarrow 3^-} \frac{x^2+1}{3+x} = \frac{+}{-} = -\infty$       and  $\lim_{x \rightarrow 3^+} \frac{x^2+1}{3+x} = \frac{+}{+} = +\infty$   
 pick "4"      pick "2"

Therefore,  $\lim_{x \rightarrow 3} \frac{x^2+1}{3+x} = \text{DNE}$

b/c the one-sided limits do not match

Examples:

Find the following limits, but be sure to check both the left and right sides!

$\lim_{x \rightarrow 1} \frac{4}{x-1}$        $\lim_{x \rightarrow 1^+} \frac{4}{x-1} = \frac{+}{+} = +\infty$       }      so  $\lim_{x \rightarrow 1} \frac{4}{x-1} = \text{DNE}$   
 $\lim_{x \rightarrow 1^-} \frac{4}{x-1} = \frac{+}{-} = -\infty$

$\lim_{x \rightarrow 1} \frac{2}{(x-1)^2}$        $\lim_{x \rightarrow 1^+} \frac{2}{(x-1)^2} = \frac{+}{+} = +\infty$       }      so  $\lim_{x \rightarrow 1} \frac{2}{(x-1)^2} = +\infty$   
 $\lim_{x \rightarrow 1^-} \frac{2}{(x-1)^2} = \frac{+}{+} = +\infty$

$\lim_{x \rightarrow 0} \frac{\sqrt{1-x}}{x^2}$        $\lim_{x \rightarrow 0^+} \frac{\sqrt{1-x}}{x^2} = \frac{+}{+} = +\infty$       }      so  $\lim_{x \rightarrow 0} \frac{\sqrt{1-x}}{x^2} = \infty$   
 $\lim_{x \rightarrow 0^-} \frac{\sqrt{1-x}}{x^2} = \frac{+}{+} = +\infty$

$\lim_{x \rightarrow 5} \frac{x^2+5x}{x^2-25}$        $\lim_{x \rightarrow 5^+} \frac{x^2+5x}{x^2-25} = \frac{+}{+} = +\infty$       }      so  $\lim_{x \rightarrow 5} \frac{x^2+5x}{x^2-25} = \text{DNE}$   
 $\lim_{x \rightarrow 5^-} \frac{x^2+5x}{x^2-25} = \frac{+}{-} = -\infty$